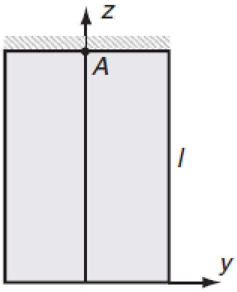
Example: Stretching of a Prismatical Bar by its own Weight

As an example of a simple direct integration problem, consider the case of a uniform prismatic bar stretched by its own weight, as shown in the figure. The body forces for this problem are:

$$F_x = F_y = 0, \quad F_z = -\rho g \tag{a}$$

where ρ is the material mass density and g is the acceleration of gravity.



Assuming that on each cross-section we have uniform tension produced by the weight of the lower portion of the bar, the stress field would take the form:

$$\sigma_x = \sigma_y = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0 \tag{b}$$

The equilibrium equations reduce to the simple result:

$$\frac{\partial \sigma_z}{\partial z} = -F_z = \rho g \tag{c}$$

This equation can be integrated directly, and applying the boundary condition $\sigma_z(z=0)=0$ gives $\sigma_z(z) = \rho g z$.

The compatibility equations are also satisfied bu the solution of (b) and $\sigma_z(z) = \rho g z$, hence it is the correct solution of the problem for a uniform distribution of the forces at the top. It coincides with the solution which is usually given in elementary strengh of materials.

Let us consider now the displacements. From Hooke's law we find:

$$\varepsilon_z = \frac{\partial w}{\partial z} = \frac{\sigma_z}{E} = \frac{\rho g z}{E} \tag{d}$$

$$\varepsilon_x = \varepsilon_y = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -\nu \frac{\rho g z}{E} \tag{e}$$

$$\tau_{xy} = \tau_{xz} = \tau_{yz} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \tag{f}$$

The discplacements u, v and w can now be found by integrating Eqs. (d), (e) and (f). Integration of Eq. (d) gives:

$$w = \frac{\rho g z^2}{2E} + w_0 \tag{g}$$

where w_0 is a function of x and y, to be determined later. Substuing (g) in the second and third of Eqs. (f), we find:

$$\frac{\partial w_0}{\partial x} + \frac{\partial u}{\partial z} = 0, \quad \frac{\partial w_0}{\partial y} + \frac{\partial v}{\partial z} = 0$$

From which

$$u = -z\frac{\partial w_0}{\partial x} + u_0, \quad v = -z\frac{\partial w_0}{\partial y} + v_0 \tag{h}$$

In which u0 and v0 are functions of x and y only. Substituing expressions (h) into Eqs. (e), we find:

$$-z\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial u_0}{\partial x} = -\nu\frac{\rho g z}{E}, \quad -z\frac{\partial^2 w_0}{\partial y^2} + \frac{\partial v_0}{\partial y} = -\nu\frac{\rho g z}{E} \tag{k}$$

Remembering that u_0 and v_0 do not depend on z, Eqs. (i) can be satisfied only if:

$$\frac{\partial u_0}{\partial x} = \frac{\partial v_0}{\partial y} = 0, \quad \frac{\partial^2 w_0}{\partial x^2} = \frac{\partial^2 w_0}{\partial y^2} = -\nu\frac{\rho g}{E} \tag{l}$$

Substituing expressions (h) for u and v into the first of Eqs. (f), we find:

$$-2z\frac{\partial^2 w_0}{\partial x \partial y} + \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}$$

And, since u_0 and v_0 do not depend on z, we must have:

$$\frac{\partial^2 w_0}{\partial x \partial y} = 0, \quad \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} = 0 \tag{m}$$

From Eqs. (l) and (m) general expresison can now be written for the function u_0, v_0 and w_0. It is easy to show that all there equations are satisfied by:

$$\begin{aligned} u_0 &= \delta y + \delta_1 \\ v_0 &= -\delta x + \gamma_1 \\ w_0 &= \frac{\nu \rho g}{2E}(x^2 + y^2) + \alpha x + \beta y + \gamma \end{aligned}$$

In which α, β, γ, δ, δ_1, γ_1 are arbitrary constants. Now, from Eqs. (g) and (h), the general expressions for the displacements are:

$$\begin{aligned}
u &= -\frac{\nu\rho g x z}{E} - \alpha z + \delta y + \delta_1 \\
v &= -\frac{\nu\rho g y z}{E} - \beta z - \delta x + \gamma_1 \\
w &= \frac{\rho g z^2}{2E} + \frac{\nu\rho g}{2E}(x^2 + y^2) + \alpha x + \beta y + \gamma
\end{aligned} \tag{n}$$

The six arbitrary constants must be determined from the conditions at the support. The support must be such as to prevent any movement of the bar as a rigid body.

To prevent a translatory motion of the bar, let us fix the centroid A of the upper end of the bar so that $u = v = w = 0$ for $x = y = 0$ and $z = l$.

To eliminate the rotation of the bar about axes through the point A, parallel to the x and y axes, let us fix an element of the z-axis at A. Then $\partial u/\partial z = \partial v/\partial z = 0$ at that point.

The possibility of rotation about z-axis is eliminated by fixing an elemental área through A. Using Eqs (n) above six conditions at the point A become:

$$\begin{aligned}
&\alpha l + \delta_1 = 0 \\
&-\beta l + \gamma_1 = 0 \\
&\frac{\rho g l^2}{2E} + \gamma = 0 \\
&\alpha = \beta = \delta = 0
\end{aligned}$$

Hence,

$$\begin{aligned}
\delta_1 &= 0 \\
\gamma_1 &= 0 \\
\gamma &= -\frac{\rho g l^2}{2E}
\end{aligned}$$

and the final expressions for the displacement are:

$$\begin{aligned}
u &= -\frac{\nu\rho g x z}{E} \\
v &= -\frac{\nu\rho g y z}{E} \\
w &= \frac{\rho g z^2}{2E} + \frac{\nu\rho g}{2E}(x^2 + y^2) - \frac{\rho g l^2}{2E}
\end{aligned}$$

It may be seen that points on the z-axis have only vertical displacements:

$$w = \frac{\rho g}{2E}(l^2 + z^2)$$

Other points of the bar, on account of lateral contraction, have not vertical but also horizontal displacements. Lines which are parallel to the z-axis before deformation become inclined to the axis after deformation, and the form of the bar after deformation.

Cross sections of the bar perpendicular to the z-axis after deformation are curved to the surface of a paraboloide. Points on the cross section $z = c$, for instance, after deformation will be on the surface

$$z = c + w = c + \frac{\rho g c^2}{2E} + \frac{\nu \rho g}{2E}(x^2 + y^2) - \frac{\rho g l^2}{2E}$$

The surface is perpendicular to all longitudinal fibers of the bar, these being inclined to the z-axis after deformation, so that there is no shearing strain γ_{xy} or γ_{xz}.